НАЦИОНАЛЬНЫЙ ИССЛЕДОВАТЕЛЬСКИЙ УНИВЕРСИТЕТ

All the stable sets I know: definitions, characterizations, relations, generalizations, interpretations

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Alternatives, preferences, choices

A – the *general set* of alternatives.

X – the *feasible set* of alternatives: $X \subseteq A \wedge X \neq \varnothing$. The feasible set is a variable.

R – *social preferences*, $R \subseteq A \times A$.

R is presumed to be complete: $\forall\, x \in A, \forall\, y \in A, (x, y) \in R \vee (y, x) \in R$.

P – *strict* social preferences, $P \subseteq R$: $(x, y) \in P \Leftrightarrow ((x, y) \in R \wedge (y, x) \notin R)$.

A *social preference-based choice correspondence* is a mapping S: $2^A \backslash \varnothing \times 2^{A \times A} \rightarrow 2^A$

with arguments X and P and values in the set of subsets of X.

It is presumed that S depends on X and P only through restriction of P on X:

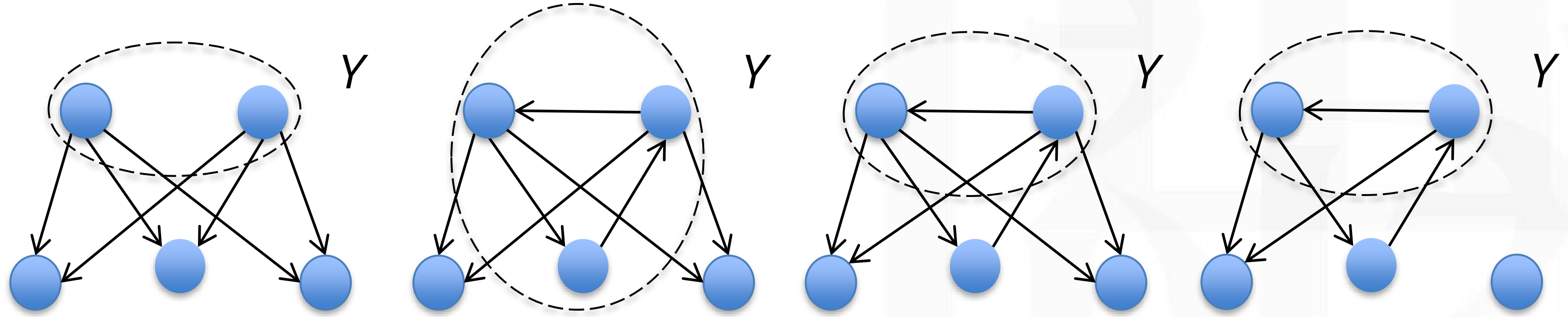
$$S = S(X, P) = S(P|_X) \subseteq X$$

i.e. social choices are dependent on social preferences for available alternatives only.

Stable sets

A nonempty subset Y of X is called

R-dominant	if	$\forall x \in X \setminus Y$,	$\forall y \in Y: yRx$
P-dominant	if	$\forall x \in X \setminus Y$,	$\forall y \in Y: yPx$
P-dominating	if	$\forall x \in X$,	$\exists y \in Y: yPx$
P-externally stable	if	$\forall x \in X \setminus Y$,	$\exists y \in Y: yPx$
R-externally stable	if	$\forall x \in X \setminus Y$,	$\exists y \in Y: yRx$
Self-protecting	if	$\forall x \in X$,	$(\exists y \in Y: yPx) \vee (\forall y \in Y, yRx)$
Weakly stable	if	$\forall x \in X \setminus Y$,	$(\exists y \in Y: yPx) \vee (\forall y \in Y, yRx)$



P-dominant *P-dominating* *P-ext. stable* *Weakly stable*

Minimal stable sets

A set *Y* is called *minimal* with respect to a given property if *Y* has the property and none of *Y*'s proper nonempty subsets does.

Tournament solutions: the union of all minimal

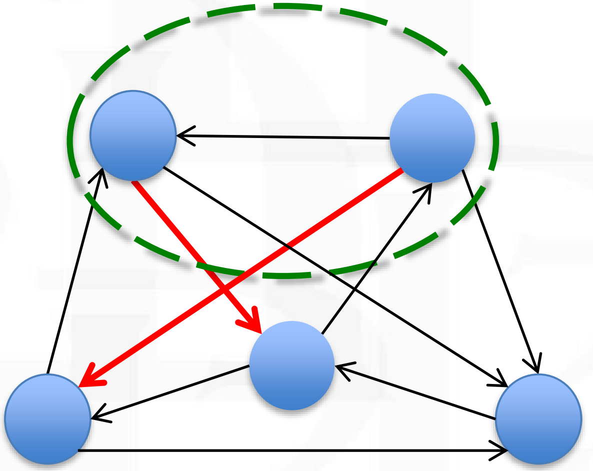
R-dominant sets	*WTC*	a.k.a. the *weak top cycle* (Good 1971, Smith 1973)
P-dominant sets	*STC*	a.k.a. the *strong top cycle* (Schwartz 1970, 1972)
P-dominating sets	*D*	(Duggan 2013, Subochev 2016)
P-externally stable sets	*ES*	(Wuffl, Feld, Owen & Grofman 1989, Subochev 2008)
R-externally stable sets	*RES*	(Aleskerov & Subochev 2009, 2013)
Self-protecting sets	*SP*	(Roth 1976, Subochev 2020)
Weakly stable sets	*WS*	(Aleskerov & Kurbanov 1999)

Cooperative game interpretation

Sets of alternatives can be interpreted as *coalitions* (e.g. sport teams, political cliques etc.). External stability guaranties a victory of a coalition (represented by its champion) in a duel with any outsider (the "*Three Musketeers*" principle). Consequently, *ES* can be viewed as a solution of the following simple cooperative game:

- X is the set of players;
- Value function $v(Y)=1$ if Y is externally stable,
  $v(Y)=0$ otherwise.

Then *ES* is the support of Banzhaf and Shapley-Shubik power indices.



Externally stable

The covering relations and the uncovered sets

The covering relations (Fishburn, 1977; Miller, 1980)

The covering relation $C(P|_X) \subseteq X^2$, is a strengthening of the strict social preferences P:

1. The Miller covering relation C_M : $x\, C_M\, y \Leftrightarrow x\, P\, y \wedge P|_X^{-1}(y) \subset P|_X^{-1}(x)$.
2. The weak Miller covering C_{WM}: $x\, C_{WM}\, y \Leftrightarrow P|_X^{-1}(y) \subset P|_X^{-1}(x)$.
3. The Fishburn covering C_F : $x\, C_F\, y \Leftrightarrow x\, P\, y \wedge P|_X(x) \subset P|_X(y)$.
4. The weak Fishburn covering C_{WF} : $x\, C_{WF}\, y \Leftrightarrow P|_X(x) \subset P|_X(y)$.

Note that $C(P|_X)$ is not a restriction of $C(P)$ on X: $C(P|_X) \not\equiv C(P) \cap X^2$.

The set of all alternatives that are not covered in X by any alternative is called ***the uncovered set*** of a feasible set X.

The set of all alternatives that are not weakly covered in X will be called ***the inner uncovered set*** of a feasible set X.

The Miller and Fishburn uncovered sets and their inner versions will be denoted UC_M and UC_F, UC_{IM} and UC_{IF}, correspondingly.

The uncovered sets and the externally stable sets

Theorem A. Suppose $|X|<\infty$. $x \in ES \Leftrightarrow \exists\, y \in UC_F\colon x\,P\,y \vee x \in UC_F$.

Corollary to Theorem A. ES is a union of UC_F and all $P(x)$ such that $x \in UC_F$

Theorem B. Suppose $|X|<\infty$. $x \in RES \Leftrightarrow \exists\, y \in UC_{IM}\colon x\,R\,y.$

Corollary to Theorem B. RES is a union of all $R(x)$ such that $x \in UC_{IM}$

Theorem C. Suppose $|X|<\infty$. $x \in D \Leftrightarrow \exists\, y \in UC_{IF}\colon x\,P\,y.$

Corollary to Theorem C. D is a union of all $P(x)$ such that $x \in UC_{IF}$

$UC_F \subseteq ES$ $\qquad$ $UC_M \subseteq RES$

D is not nested with the UC even when P is a tournament.

Relation to other solutions (Tournaments)

	C	*SL*	*B*	*MC*	*BP*	*UC*	*D*	*ES*	*UCp*	*TC*
D	Æ	É	É	É	É	Ç	=	Ì	Ì	Ì
ES	É	É	É	É	É	É	É	=	Ì	Ì

Relation to other solutions (General case)

	UC_{IM}	UC_{M}	UC_{IF}	UC_{F}	UC_{McK}	UC_{D}	*D*	*WS*	*ES*	*RES*	*UCp*	*STC*	*UT*	*WTC*
D	Æ	Ç	Æ	Æ	Ç	Ç	=	Ç	Ì	Ç	Ì	Ç	Ç	Ì
WS	Æ	Ç	Æ	Ç	Ç	Ç	Ç	=	Ç	Ç	Ì	Ç	Ì	Ì
ES	Æ	Ç	É	É	Ç	Ç	É	Ç	=	Ç	Ì	Ç	Ç	Ì
RES	É	É	Ç	Ç	Ç	Ç	Ç	Ç	Ç	=	Ç	Ç	Ç	Ì

Infinity of alternatives. Problems and (counter)examples

1. The uncovered set may be empty.
2. An externally stable set may not contain a minimal externally stable subset. The same is true for dominating sets.
3. The propositions of Theorems **A**, **B** and **C** may not hold.

Example 1: X - an infinite sequence $\{x_n\}$, n=1, 2, 3,…
$x_n P x_m \Leftrightarrow n>m$. P is a tournament and a linear order.
Evidently, $x_n C x_m \Leftrightarrow n>m$. Consequently, $UC=\varnothing$.
Any infinite subset of X is dominating and, consequently, externally stable.
Any finite subset of X is not externally stable and, consequently, dominating.
Since any infinite set always includes a proper subset which is infinite, there is no minimal dominating or externally stable set.

Example 2: X - three infinite sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$, n=1, 2, 3,…
$x_n P y_m \Leftrightarrow n>m$; $y_n P y_m \Leftrightarrow n>m$; $z_n P z_m \Leftrightarrow n>m$; $\forall n$, $\forall m$, $x_n P y_m \wedge y_n P z_m \wedge z_n P x_m$.

Then $UC=\varnothing$, but any triplet $\{x_n, y_m, z_k\}$ is a minimal dominating and a minimal externally stable set; consequently, $D=ES=X$, which contradicts all three theorems.

Infinity of alternatives. Positive results

Proposition: Suppose there is a topology Ω such that
$R(x) \cap X$ is compact in Ω for all $x \in X$.
Then $UC_{IF} \neq \varnothing$

(Banks, Duggan & Le Breton 2006)

Lemma: Suppose there is a topology Ω such that
X is compact in Ω, and $P^{-1}(x) \cap X$ is open for any $x \in X$.
Then any dominating set contains a finite dominating subset.

Consequently, $D \neq \varnothing$. Consequently, $WS \neq \varnothing$ and $RES \neq \varnothing$.
Additionally, if the core is either empty or P-externally stable then $ES \neq \varnothing$.

Theorem D (Generalization of Theorem **C):**

Suppose there is a topology Ω such that
X is compact in Ω, and $P^{-1}(x) \cap X$ is open for any $x \in X$.
Then $x \in D \Leftrightarrow \exists\, y \in UC_{IF}: x\, P\, y$.

Thank you!